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Embeddings of κ-metrizable spaces into function spaces

Masami Sakai

Department of Mathematics, Kanagawa University, Yokohama, 221, Japan

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Abstract

This paper is motivated by V.V. Uspenskii's results on embeddings of spaces into function spaces and the author's results on countable κ-metrizable spaces. For a Tychonoff topological space Y we denote by $C_p(Y)$ the space of all real-valued continuous functions on Y with the topology of pointwise convergence. In this paper, we are interested in an "intrinsic" characterization of spaces which can be embedded into $C_p(Y)$ on some compact space Y, and an estimation of the number of countable stratifiable κ-metrizable spaces. We prove that (1) if X is a space with a unique nonisolated point, and the nonisolated point is a G_δ-point in X, then X can be embedded into $C_p(Y)$ for some compact space Y iff X is κ-metrizable in the sense of E.V. Ščepin, (2) the number of countable stratifiable κ-metrizable spaces is 2^ω. As an application, we negatively answer a question posed by A.V. Arkhangel'skii.

Keywords: Eberlein–Grothendieck space; Monotonically normal space; Stratifiable space; κ-metrizable space

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1. Introduction

In this paper by a space we shall always mean a Tychonoff topological space. The letter $\mathbb{N}$ denotes the positive integers with the discrete topology, and ω (respectively ω_1) is the first infinite (respectively first uncountable) ordinal. Further, $D = \{0, 1\}$ is the discrete two-point space and $\mathbb{R}$ is the real line. Unexplained notions and terminology are the same as in [4].

For a space Y we denote by $C_p(Y)$ the space of all real-valued continuous functions on Y with the topology of pointwise convergence. A basic open set of $C_p(Y)$ is of the form $[y_1, \ldots, y_n; U_1, \ldots, U_n] = \{f \in C_p(Y) : f(y_i) \in U_i,\ i = 1, \ldots, n\}$, where $n \in \mathbb{N}$, $y_i \in Y$ and each U_i is an open subset of the real line $\mathbb{R}$. It is easy to see that $C_p(Y)$ is

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dense in $\mathbb{R}^Y$. Many interesting results on $C_p(Y)$ are known, we refer to the book [3] as a survey.

In the second section, we are concerned with an "intrinsic" characterization of spaces which can be embedded into $C_p(Y)$ on some compact space Y, see [3, Problem III.1.9]. It is known that if Y is compact, $C_p(Y)$ has rich properties. For example, if Y is compact, more generally if each finite power of Y is Lindelöf, then $C_p(Y)$ has countable tightness [3, Theorem II.1.1], where a space X is said to have countable tightness if for any $A \subset X$ and any point $x \in X$ such that $x \in \bar{A}$, there is a set $B \subset A$ for which $|B| \leqslant \omega$ and $x \in \bar{B}$. This fact implies that if Y is compact, then the topology of $C_p(Y)$ is determined by the collection of countable subsets of $C_p(Y)$. Hence the following question seems to be natural and important: what spaces, especially what countable spaces can be embedded into $C_p(Y)$ where Y is compact [1, p. 90; 3, p. 92]? For this question, Uspenskii obtained the following Theorems 1.1 and 1.2. According to [19], if $p \in X$, we denote by $W_X(p)$ the subspace $\{f \in D^X\colon\ p \in \operatorname{Int} f^{\leftarrow}(0)\}$ of D^X. For convenience, we call a space X an Eberlein–Grothendieck space (abbreviation an EG-space) [3] if it can be embedded into $C_p(Y)$ for some compact space Y. Every metrizable space is an EG-space [1, Theorem 4.4.2].

Theorem 1.1 [19, Theorem 2]. *If X is an EG-space and $p \in X$, then $W_X(p)$ is σ-compact.*

The converse of Theorem 1.1 is true for some class of spaces.

Theorem 1.2 [19, Proposition 3]. *If X is a space with a unique nonisolated point p, and $W_X(p)$ is σ-compact, then X is an EG-space.*

By making use of Theorem 1.1, Uspenskii gave two countable non-EG-spaces. One of them is a subspace $\mathbb{N} \cup \{p\}$ of the Stone–Čech compactification $\beta\mathbb{N}$, where $p \in \beta\mathbb{N} - \mathbb{N}$ [19, Theorem 3]. And the other is the sequential fan $S(\omega)$ with ω converging sequences [19, Example 3]. The space $S(\omega)$ is the quotient space obtained from the topological sum of ω converging sequences by identifying limit points.

We would like to show that κ-metrizability in the sense of Ščepin fits an "intrinsic" characterization of EG-spaces. Let X be a space with a unique nonisolated point p and let p a G_δ-point in X, then we prove that X is an EG-space iff X is κ-metrizable. It is known that neither $\mathbb{N} \cup \{p\}$ nor $S(\omega)$ is κ-metrizable. As an application, we negatively answer a question posed by A.V. Arkhangel'skii.

In the third section, we are concerned with an estimation of the number of countable stratifiable κ-metrizable spaces. It is well known that the number of countable metrizable spaces is 2^ω [9]. Recall that the space of rational numbers is a universal space for all countable metrizable spaces [4, 4.3.H.(b)]. It is not difficult to see that both the number of countable stratifiable spaces and the number of countable κ-metrizable spaces are more than 2^ω. We prove that the number of countable stratifiable κ-metrizable spaces is 2^ω.

2. Embeddings into function spaces

We begin with the definition of κ-metrizable spaces. A subset R of a space X is called a regular closed set if it is the closure of an open subset of X. According to [14], a κ-metric on a space X is a nonnegative real-valued function $\rho(x, R)$ of two variables, a point $x \in X$ and a regular closed subset R in X, satisfying the following requirements:

(K1) $\rho(x, R) = 0$ iff $x \in R$;

(K2) if $R_1 \subset R_2$, then $\rho(x, R_1) \geqslant \rho(x, R_2)$ for every $x \in X$;

(K3) $\rho(x, R)$ is continuous with respect to x for every R;

(K4) $\rho(x, \overline{\bigcup_\alpha R_\alpha}) = \inf_\alpha \rho(x, R_\alpha)$ for every increasing transfinite sequence $\{R_\alpha\colon \alpha < \tau\}$.

A space on which there exists a κ-metric is called κ-metrizable. Obviously every metrizable space is κ-metrizable. The product of any collection of κ-metrizable spaces is κ-metrizable [14, Theorem 2]. Not every subspace of a κ-metrizable space is κ-metrizable, because any product of the real lines is κ-metrizable. But both dense subspaces and regular closed subspaces of a κ-metrizable space are κ-metrizable. For any space Y, since $C_p(Y)$ is dense in $\mathbb{R}^Y$, $C_p(Y)$ is κ-metrizable.

In [13] we gave a characterization of κ-metrizability for a class of spaces. For a set Y we denote by $P(Y)$ the set of all subsets of Y. We set $[1, \omega] = \mathbb{N} \cup \{\omega\}$.

Theorem 2.1 [13]. *Let $X = Y \cup \{p\}$ be a space such that every point of Y is isolated in X and the specific point p is a G_δ-point in X. Then the following are equivalent:*

(a) *X is κ-metrizable,*

(b) *there exists a function $\varphi\colon P(Y) \to [1, \omega]$ with the following conditions;*

(1) *for any $F \in P(Y)$, F is closed in X iff $\varphi(F) < \omega$;*

(2) *if $F_1 \subset F_2$, then $\varphi(F_1) \leqslant \varphi(F_2)$;*

(3) *if $\{F_\alpha\colon \alpha < \tau\}$ is an increasing collection in $P(Y)$ such that $\varphi(F_\alpha) = k < \omega$ for every $\alpha < \tau$, then $\varphi(\bigcup_\alpha F_\alpha) = k$.*

We return to Theorem 1.1 due to Uspenskii. Theorem 1.1 claims that if X is an EG-space and $p \in X$, then there is a sequence $\{K_n\colon n \in \mathbb{N}\}$ of compact subsets in D^X such that $W_X(p) = \bigcup\{K_n\colon n \in \mathbb{N}\}$. Below we need a specific property of K_n. For completeness, we recall the construction of K_n in accordance with Uspenskii's proof. Let $\varphi\colon X \to C_p(Y)$ be an embedding, where Y is compact. For each $y \in Y$ we set $F_y = \{x \in X\colon |\varphi(x)(y) - \varphi(p)(y)| \geqslant 1/n\}$. For each $n \in \mathbb{N}$ we define a mapping $h_n\colon Y \to W_X(p)$ by assigning to each $y \in Y$ the characteristic function $h_n(y)\colon X \to D$ of F_y in X. Exactly speaking, $h_n(y)(x) = 1$ iff $|\varphi(x)(y) - \varphi(p)(y)| \geqslant 1/n$. Then $\overline{h_n(Y)} \subset W_X(p)$ for each $n \in \mathbb{N}$, where the closure is taken in D^X. We set, in order, $A(n) = \overline{h_n(Y)}$, $A(n,k) = \{\sup\{f_1, \ldots, f_k\}\colon f_i \in A(n)\}$ and $B(n,k) = \{f \in D^X\colon f \leqslant g \text{ for some } g \in A(n,k)\}$. Then we can see that each $B(n,k)$ is compact and $W_X(p) = \bigcup\{B(n,k)\colon n, k \in \mathbb{N}\}$. We enumerate $\{B(n,k)\colon n, k \in \mathbb{N}\} = \{K_n\colon n \in \mathbb{N}\}$. The property of K_n we use later is the following: if $f, g \in D^X$ and $f \leqslant g \in K_n$, then $f \in K_n$.

For a space X and $p \in X$, we denote by $X(p)$ the space obtained by making every point but p isolated. Every point distinct from p is isolated in $X(p)$ and a neighborhood of p in $X(p)$ is the same as that in X.

Lemma 2.2. *Let X be a space and p a G_δ-point in X. If X is an EG-space, then $X(p)$ is κ-metrizable.*

Proof. We set $X(p) = Y \cup \{p\}$, where $Y = X - \{p\}$. By Theorem 1.1 and the statement above, there exists a sequence $\{K_n\colon n \in \mathbb{N}\}$ of compact subsets in D^X such that (1) $W_X(p) = \bigcup\{K_n\colon n \in \mathbb{N}\}$, (2) if $f, g \in D^X$ and $f \leqslant g \in K_n$, then $f \in K_n$. For each $A \subset Y$, let $f_A : X \to D$ be the characteristic function on A in X. If $f_A \in W_X(p)$, then we set $k(A) = \min\{n \in \mathbb{N}\colon f_A \in K_n\}$. To show that $X(p)$ is k-metrizable, we define a function $\varphi : P(Y) \to [1, \omega]$ in the following manner:

$$\varphi(A) = \begin{cases} \omega, & \text{if } f_A \in D^X - W_X(p), \\ k(A), & \text{if } f_A \in W_X(p). \end{cases}$$

We examine the conditions in Theorem 2.1. Suppose that $A \subset Y$ is closed in $X(p)$. Then there is an open neighborhood U of p in X such that $A \cap U = \emptyset$. Hence $p \in U \subset X - A = f_A^{\leftarrow}(0)$, which implies $f_A \in W_X(p)$. Thus $\varphi(A) = k(A) < \omega$. Conversely suppose $\varphi(A) < \omega$. Then $f_A \in W_X(p)$, which implies that there is an open neighborhood U of p in X such that $p \in U \subset f_A^{\leftarrow}(0) = X - A$. Hence A is closed in $X(p)$. Next suppose $A \subset B \subset Y$ and $\varphi(B) = n < \omega$. Then $f_A \leqslant f_B \in K_n$, hence $f_A \in K_n$. Thus $\varphi(A) \leqslant n$. Last let $\{A_\alpha\colon \alpha < \tau\}$ be an increasing collection of subsets in Y such that $\varphi(A_\alpha) = n < \omega$ for each α. We set $A = \overline{\bigcup\{A_\alpha\colon \alpha < \tau\}}$. Since the collection is increasing, it is not difficult to see that $f_A \in \overline{\{f_{A_\alpha}\colon \alpha < \tau\}} \subset K_n$. Consequently $\varphi(A) = n$. This completes the proof. □

The condition "p a G_δ-point in X" in Lemma 2.2 is essential. Let $D(\omega_1)$ be the discrete space of cardinality ω_1. Let $X = D(\omega_1) \cup \{p\}$ be the one-point compactification of $D(\omega_1)$. It is known that X can be embedded into $C_p(X)$, for example see [3, Proposition III.3.2]. But $X(p) = X$ is not κ-metrizable, because of [14, Theorem 6].

A space X is said to have countable fan tightness [2] if for each $x \in X$ and each sequence $\{A_n\colon n \in \mathbb{N}\}$ of subsets of X such that $x \in \bigcap\{\bar{A}_n\colon n \in \mathbb{N}\}$, there are finite sets $B_n \subset A_n$ such that $x \in \overline{\bigcup\{B_n\colon n \in \mathbb{N}\}}$. A space with countable fan tightness has countable tightness. Every EG-space has countable fan tightness [3, Theorem III.1.1].

Lemma 2.3 [13]. *Every monotonically normal κ-metrizable space has countable fan tightness, in particular it has countable tightness.*

We give the definition of monotonically normal spaces in the next section. We note here only that a space with a unique nonisolated point is monotonically normal.

Lemma 2.4. *Let $X = Y \cup \{p\}$ be a space such that every $y \in Y$ is isolated in X. If X is κ-metrizable, then X is an EG-space.*

Proof. By Theorem 1.2 we have only to show that $W_X(p) = \{f \in D^X\colon p \in \operatorname{Int} f^{\leftarrow}(0)\}$ is σ-compact. Since X is κ-metrizable, there exists a function $\varphi : P(Y) \to [1, \omega]$ with the properties in Theorem 2.1(b). We note that the condition "p is a G_δ-point" in Theorem 2.1 is necessary only for the implication (b) $\to$ (a). For each $A \subset Y$, let $f_A : X \to D$ be the characteristic function on A in X. We set $K_n = \{f_A\colon A \subset Y,\ \varphi(A) \leqslant n\}$ for each $n \in \mathbb{N}$. Then $K_n \subset W_X(p)$ is obvious. We show $\overline{K}_n \subset W_X(p)$, where the closure is taken in D^X. Fix $n \in \mathbb{N}$ and assume that $g \in \overline{K}_n - W_X(p)$. Then $p \in \overline{g^{\leftarrow}(1)}$ (note that $g^{\leftarrow}(1) \subset Y$, because of $g \in \overline{K}_n \subset \overline{W_X(p)} \subset \{f \in D^X\colon f(p) = 0\}$). By Lemma 2.3 there is a countable subset $Z = \{z_n\colon n \in \mathbb{N}\}$ of $g^{\leftarrow}(1)$ such that $p \in \bar{Z}$. We set $Z_k = \{z_1, \ldots, z_k\}$ for each $k \in \mathbb{N}$. Since the set $U_k = \{f \in D^X\colon f(z_i) = 1,\ i = 1, \ldots, k\}$ is an open neighborhood of g in D^X, we can find an $f_A \in K_n \cap U_k$, which implies $Z_k \subset A$. Hence $\varphi(Z_k) \leqslant \varphi(A) \leqslant n$. Since $\{Z_k\colon k \in \mathbb{N}\}$ is increasing, $\varphi(Z) = \varphi(\bigcup\{Z_k\colon k \in \mathbb{N}\}) \leqslant n$. It follows that Z is closed in X. This is a contradiction to $p \in \bar{Z}$. Therefore we obtain $\overline{K}_n \subset W_X(p)$ for each $n \in \mathbb{N}$. Conversely suppose $g \in W_X(p)$. Then $g^{\leftarrow}(1)$ is closed in X. Hence $\varphi(g^{\leftarrow}(1)) \leqslant n$ for some $n \in \mathbb{N}$. If we set $A = g \leftarrow (1)$, then $g = f_A \in K_n$. Consequently $W_X(p) = \bigcup\{\overline{K}_n\colon n \in \mathbb{N}\}$. This completes the proof. $\square$

By Lemma 2.2 and Lemma 2.4, we immediately get the following theorem which gives an "intrinsic" characterization of EG-spaces for a class of spaces.

Theorem 2.5. *Let X be a space with a unique nonisolated point, and the specific point is a G_δ-point in X. Then X is an EG-space iff X is κ-metrizable.*

If a countable space is an EG-space, then it can be embedded into $C_p(C)$, where C is the Cantor set [3, p. 95].

Corollary 2.6. *Let X be a countable space with a unique nonisolated point. Then the following are equivalent.*

(1) *X is an EG-space,*
(2) *X can be embedded into $C_p(C)$, where C is the Cantor set,*
(3) *X is κ-metrizable.*

Each subspace $\mathbb{N} \cup \{p\}$ of $\beta\mathbb{N}$ is not κ-metrizable, because an extremally disconnected κ-metrizable space is discrete [14, Theorem 14]. The sequential fan $S(\omega)$ is not also κ-metrizable, because a Lašnev space (i.e. a continuous closed image of a metrizable space) is metrizable if it is κ-metrizable [16, Theorem 2.2]. Therefore, as Uspenskii already proved, neither $\mathbb{N} \cup \{p\}$ nor $S(\omega)$ is an EG-space by Corollary 2.6.

Let $S_2 = (\mathbb{N} \times \mathbb{N}) \cup \mathbb{N} \cup \{p\}$ be the Arens' space [4, Example 1.6.19]. Each point of $\mathbb{N} \times \mathbb{N}$ is isolated and a basic open neighborhood of $n \in \mathbb{N}$ is of form $\{n\} \cup \{(m, n)\colon m \geqslant k\}$. A set U is a neighborhood of p iff $p \in U$ and U is a neighborhood of all but finitely many $n \in \mathbb{N}$. It is obvious that the fan tightness of S_2 is uncountable. Hence S_2 is not an EG-space.

A space X is said to be dominated by a closed cover $\mathcal{F}$ if whenever a subset A of X has a closed intersection with every element of some $\mathcal{G} \subset \mathcal{F}$ which covers A, then A is closed in X. The space S_2 is dominated by a closed cover of metric subsets. A Lašnev space is not always dominated by a closed cover of metric subsets [17, Example 1.5].

In general we have:

Proposition 2.7. *The following statements hold:*

(1) *an EG-space is metrizable if it is Lašnev or dominated by a closed cover of metric subsets,*

(2) *an extremally disconnected EG-space is discrete.*

Proof. (1) Let X be an EG-space. A nonmetrizable Lašnev space contains a copy of $S(\omega)$, see the proof of [16, Theorem 2.2]. Hence, if X is a Lašnev space, then X is metrizable. If X is dominated by a closed cover of metric subsets, as described in the proof of Theorem 3.6 in [15], X is sequential and hereditarily normal. Since X is an EG-space, X contains no copy of $S(\omega)$ and no copy of S_2. Hence X is strongly Fréchet by Lemma 3.1 in [15]. Furthermore, by Lemma 3.5 in [15], X is metrizable.

(2) Let X be an extremally disconnected EG-space. Assume that a point $p \in X$ is nonisolated. Since X has countable tightness, there exists a disjoint collection $\{U_n\colon n \in \mathbb{N}\}$ of clopen sets in X such that $p \in \overline{\bigcup\{U_n\colon n \in \mathbb{N}\}} - \bigcup\{U_n\colon n \in \mathbb{N}\}$. We set $Y = \{p\} \cup (\bigcup\{U_n\colon n \in \mathbb{N}\})$. The space Y is extremally disconnected and p is a G_δ-point in Y. By Lemma 2.2, $Y(p)$ is κ-metrizable. We set $q = \{A\colon A \subset \mathbb{N} \text{ and } p \in \overline{\bigcup\{U_n\colon n \in A\}}\}$. It is easy to see that q is a free ultrafilter on $\mathbb{N}$. Let $\varphi : Y(p) \to \mathbb{N} \cup \{q\} \subset \beta\mathbb{N}$ be the continuous map defined by $\varphi(p) = q$ and $\varphi(x) = n$ if $x \in U_n$ for each $n \in \mathbb{N}$. By Theorem 14 in [14], we see that $\{p\} = \varphi^{\leftarrow}(q)$ is open in $Y(p)$. This is a contradiction. Thus X is discrete. □

By Corollary 2.6 we have the following corollary which will be used in the next section. Note that the cardinality of $C_p(C)$ is 2^ω.

Corollary 2.8. *The number of nonhomeomorphic countable κ-metrizable spaces with a unique nonisolated point is at most 2^ω.*

Now we conclude this section with an application.

A space X is called bisequential [10] if every ultrafilter converging to a point $x \in X$ contains a decreasing sequence $\{A_n\colon n \in \mathbb{N}\}$ converging to x. Every first countable space is bisequential, and every bisequential space is Fréchet. Since a countable, first countable space is metrizable, it is an EG-space [1, Theorem 4.4.2]. Arkhangel'skii asked whether every countable bisequential space is an EG-space [1, p. 90]. We give a counterexample for the question. For the purpose we recall the Cantor tree. Let T be the set of all finite sequences of 0's and 1's with the extension order $\subset$, and let C the set of all sequences of 0's and 1's whose domain is ω. In other words, C is the Cantor set $\{0,1\}^\omega$ and $T = \{f|n\colon n \in \omega,\ f \in C\}$, where $f|n$ is the restriction of f to the domain n. We give $T \cup C$ the interval topology. Exactly speaking, every point of T is isolated and a neighborhood of a point $f \in C$ is of form $\{f\} \cup J$, where J is a cofinite subset of $\{f|n\colon n \in \omega\}$. $T \cup C$ is called the Cantor tree. For each subset $K \subset C$, since the subspace $T \cup K$ of $T \cup C$ is locally compact, it has the one-point compactification $T \cup K \cup \{p\}$. We set $S(K) = T \cup \{p\}$, the subspace of $T \cup K \cup \{p\}$. By the idea of P. Nyikos, $S(K)$ is bisequential. Exactly speaking, since $T \cup K$ has a coarser Euclidean topology [12], it is hereditarily realcompact [5, Corollary 8.18]. Therefore $T \cup K \cup \{p\}$ is also hereditarily realcompact. Since every compact, hereditarily realcompact space is bisequential [11] and bisequentiality is a hereditary property, $S(K)$ is bisequential. In [13] it is proved that $S(K)$ is κ-metrizable iff K is σ-compact. So, if H is the set of irrational numbers in C, then $S(H)$ is not κ-metrizable. By Corollary 2.6, $S(H)$ is not an EG-space though it is a countable bisequential space. Thus the answer is in the negative.

3. Countable stratifiable κ-metrizable spaces

We begin with the definitions of monotonically normal spaces and stratifiable spaces, see [6] for details. A space X is called monotonically normal if for each pair (x, U) consisting of a point $x \in X$ and its neighborhood U, there exists an open set $H(x, U)$ satisfying two conditions, (a) $x \in H(x, U) \subset U$, (b) if $H(x, U) \cap H(y, V) \neq \emptyset$, then either $x \in V$ or $y \in U$. We call H a monotone operator for X. Monotone normality is a hereditary property. A space X is called stratifiable if there exists a function G which assigns to each $n \in \mathbb{N}$ and a closed set H in X, an open set $G(n, H)$ containing H such that (a) $H = \bigcap\{\overline{G(n, H)}\colon n \in \mathbb{N}\}$, (b) if $H \subset K$, then $G(n, H) \subset G(n, K)$. Every metrizable space is stratifiable, and every stratifiable space is monotonically normal [6, Theorem 5.16]. We note that a countable monotonically normal space is stratifiable. Not every countable space is monotonically normal. Heath pointed out the following fact.

Fact 3.1 [7]. *If X is a countable dense subset of $\mathbb{R}^\alpha$, where $\omega_1 \leqslant \alpha \leqslant 2^\omega$, then X is not monotonically normal.*

We obtain from the fact above the following corollary, which means that we cannot find nonmetrizable, stratifiable κ-metrizable spaces among $C_p(Y)$. Recall that every $C_p(Y)$ is κ-metrizable, and note that $C_p(Y)$ is metrizable iff Y is countable [3, Theorem I.1.1].

Corollary 3.2. *For any space Y, $C_p(Y)$ is monotonically normal iff Y is countable.*

Proof. If Y is countable, $C_p(Y)$ is metrizable. Hence $C_p(Y)$ is monotonically normal.

Conversely assume that $C_p(Y)$ is monotonically normal. Fix any $y_0 \in Y$ and put $C_p(Y, y_0) = \{f \in C_p(Y)\colon f(y_0) = 0\}$. It is easy to see that $C_p(Y)$ is homeomorphic to $\mathbb{R} \times C_p(Y, y_0)$ by the mapping $f \to (f(y_0), f - f(y_0))$. Hence $\mathbb{R} \times C_p(Y, y_0)$ is monotonically normal. By [6, Theorem 5.22], $C_p(Y, y_0)$ is stratifiable. So $C_p(Y)$ is stratifiable [6, Theorem 5.10]. Recall that a stratifiable space is a σ-space (i.e., a space with a σ-discrete closed network) [6, Theorem 5.9]. Since a monotonically normal space is collectionwise normal [6, Theorem 5.18] and the Suslin number of $C_p(Y)$ is countable

[3, Corollary 0.3.7], $C_p(Y)$ is a space with a countable network. Hence Y also has a countable network [3, Theorem I.1.3]. Suppose that Y is uncountable. Since the cardinality of a space with a countable network is at most 2^ω, $\omega_1 \leqslant |Y| \leqslant 2^\omega$. Let X be a countable dense subset of $C_p(Y)$, such a set exists because of a countable network. Since $C_p(Y)$ is monotonically normal, so is X. But this is a contradiction to Fact 3.1. We conclude that Y is countable. □

Now we would like to estimate the number of countable stratifiable κ-metrizable spaces. The number of countable metrizable spaces is 2^ω as described in the introduction. On the other hand, both the number of countable stratifiable spaces and the number of countable κ-metrizable spaces are more than 2^ω.

Fact 3.3 (cf. [18]). *There are nonhomeomorphic 2^{2^ω} countable stratifiable spaces.*

Proof. Let $\tau = 2^\omega$. Let $\{p_\alpha\colon \alpha < 2^\tau\}$ be a collection of free ultrafilters on $\mathbb{N}$ such that p_α and p_β have different types if $\alpha \neq \beta$ [20, p. 89]. The collection $\{\mathbb{N} \cup \{p_\alpha\}\colon \alpha < 2^\tau\}$ of subspaces of $\beta\mathbb{N}$ is a desired one. □

Fact 3.4. *There are nonhomeomorphic 2^{2^ω} countable κ-metrizable spaces.*

Proof. Let $\tau = 2^\omega$. Recall that a dense subset of a κ-metrizable space is κ-metrizable. Therefore we have only to find nonhomeomorphic 2^τ countable dense subsets of D^τ. Fix an embedding of $\beta\mathbb{N}$ into D^τ. Note that $\beta\mathbb{N}$ is a nowhere dense closed subset of D^τ. We fix a countable dense subset G of D^τ with $G \subset D^\tau - \beta\mathbb{N}$. Let $\{p_\alpha\colon \alpha < 2^\tau\}$ be a collection of free ultrafilters on $\mathbb{N}$ such that p_α and p_β have different types if $\alpha \neq \beta$. Since $\mathbb{N} \cup G$ is countable, without loss of generality, we may assume that each $\mathbb{N} \cup \{p_\alpha\}$ cannot be embedded into $\mathbb{N} \cup G$. We set $Y_\alpha = G \cup \mathbb{N} \cup \{p_\alpha\}$ for each $\alpha < 2^\tau$. For each $\alpha < 2^\tau$ we set $\langle\alpha\rangle = \{\beta\colon \beta < 2^\tau, Y_\alpha \text{ and } Y_\beta \text{ are homeomorphic}\}$. If φ is a homeomorphism of Y_α onto Y_β, then $\varphi(p_\alpha) = p_\beta$. In fact, if $\varphi(p_\alpha) \in \mathbb{N} \cup G$, then $\mathbb{N} \cup G$ must have a subspace which is homeomorphic to $\mathbb{N} \cup \{p_\alpha\}$. This is a contradiction. Suppose that for some α the cardinality of $\langle\alpha\rangle$ is more than τ. Since the number of bijections on $G \cup \mathbb{N}$ is τ, there exist distinct $\beta, \gamma \in \langle\alpha\rangle$ and homeomorphisms $\varphi_\beta : Y_\alpha \to Y_\beta$ and $\varphi_\gamma : Y_\alpha \to Y_\gamma$ such that $\varphi_\beta | G \cup \mathbb{N} = \varphi_\gamma | G \cup \mathbb{N}$. Hence $\varphi_\gamma \circ \varphi_\beta^{\leftarrow} : Y_\beta \to Y_\gamma$ is the homeomorphism such that the restriction to $G \cup \mathbb{N}$ is the identity map. This implies that $\mathbb{N} \cup \{p_\beta\}$ and $\mathbb{N} \cup \{p_\gamma\}$ are homeomorphic, which is a contradiction. Consequently the cardinality of $\langle\alpha\rangle$ is not more than τ. Thus we can find an $A \subset 2^\tau$ such that $|A| = 2^\tau$ and Y_α and Y_β are not homeomorphic for any distinct $\alpha, \beta \in A$. The collection $\{Y_\alpha\colon \alpha \in A\}$ of 2^{2^ω} spaces is a desired one. □

Lemma 3.5. *If X is a monotonically normal κ-metrizable space with a G_δ-point $p \in X$, then $X(p)$ is κ-metrizable.*

Proof. Let H be a monotone operator for X and ρ a κ-metric on X. Put $Y = X - \{p\}$. For each $y \in Y$ we set $U_y = H(y, Y)$. For each $A \subset Y$ we set $R(A) = \overline{\bigcup\{U_y\colon y \in A\}}$, where the closure is taken in X. Each $R(A)$ is a regular closed set in X. It is easy to see that if a subset A of Y is closed in $X(p)$, then $p \in X - R(A)$ by virtue of the monotone operator H. Hence $\rho(p, R(A)) > 0$. Let $E_1 = [1, \infty)$ and $E_n = [1/n, 1/n-1)$ for $n \geqslant 2$. Now, to show κ-metrizability of $X(p)$ by Theorem 2.1, we define a function $\varphi: P(Y) \to [1, \omega]$ in the following manner:

$$\varphi(A) = \begin{cases} k, & \text{if } A \text{ is closed in } X(p) \text{ and } \rho(p, R(A)) \in E_k, \\ \omega, & \text{if } A \text{ is not closed in } X(p). \end{cases}$$

The conditions (1) and (2) in Theorem 2.1 are easy to check. We examine (3). Let $\{A_\alpha: \alpha < \tau\}$ be an increasing collection of subsets of Y such that $\varphi(A_\alpha) = k < \omega$ for each $\alpha < \tau$. We set $A = \bigcup\{A_\alpha: \alpha < \tau\}$. Then

$$\begin{aligned} \rho(p, R(A)) &= \rho\left(p, \overline{\bigcup\{U_y: y \in A\}}\right) = \rho\left(p, \overline{\bigcup\{R(A_\alpha): \alpha < \tau\}}\right) \\ &= \inf_\alpha \rho(p, R(A_\alpha)) \in E_k. \end{aligned}$$

This implies that A is closed in $X(p)$ and $\varphi(A) = k$. Thus $X(p)$ is κ-metrizable. □

Corollary 3.6. *Let X be a countable stratifiable κ-metrizable space. Then $X(p)$ is κ-metrizable for any $p \in X$.*

The author does not know if a countable stratifiable κ-metrizable space is an EG-space. If it is true, then the corollary above is direct from Lemma 2.2.

Theorem 3.7. *The number of nonhomeomorphic countable stratifiable κ-metrizable spaces is 2^ω.*

Proof. Let $X = \{x_n: n \in \mathbb{N}\}$ be a countable set and let $\tau = (2^\omega)^+$, the successor cardinal of 2^ω. Suppose that $\{\mathcal{T}_\alpha: \alpha < \tau\}$ is a collection of topologies on X such that each $X_\alpha = (X, \mathcal{T}_\alpha)$ is stratifiable and κ-metrizable. For each $n \in \mathbb{N}$ and $\alpha < \tau$ we set $X(\alpha, n) = X_\alpha(x_n)$. By Corollary 3.6, each $X(\alpha, n)$ is κ-metrizable. We consider all possible sequences $(X(\alpha, 1), X(\alpha, 2), \ldots)$, $\alpha < \tau$. By Corollary 2.8, there are at most 2^ω sequences of this form. Therefore there exist an $A \subset \tau$ and a sequence $\{Y_n: n \in \mathbb{N}\}$ of countable κ-metrizable spaces with at most one nonisolated point such that $|A| = \tau$ and for each $n \in \mathbb{N}$ and $\alpha \in A$, $X(\alpha, n)$ is homeomorphic to Y_n. Let $\varphi_\alpha^n: Y_n \to X(\alpha, n)$ be a homeomorphism. Since the number of bijections on a countable set is 2^ω, there exist a $B \subset A$ and a sequence $\{\varphi_n: Y_n \to X: n \in \mathbb{N}\}$ of bijections such that $|B| = \tau$ and for each $n \in \mathbb{N}$ and $\alpha \in B$, $\varphi_n: Y_n \to X(\alpha, n)$ is a homeomorphism. Choose distinct $\alpha, \beta \in B$. Then for each $n \in \mathbb{N}$, $X(\alpha, n)$ and $X(\beta, n)$ are homeomorphic by the identity map on X. This implies that X_α and X_β are homeomorphic by the identity map (i.e., $\mathcal{T}_\alpha = \mathcal{T}_\beta$). This completes the proof. □

The theorem above suggests the existence of a universal space for all countable stratifiable κ-metrizable spaces. But the author does not know if such a universal space exists.

Remark 3.8. We cannot omit the condition "stratifiable" from Corollary 3.6. In fact there is a countable κ-metrizable space X and a point $p \in X$ such that $X(p)$ is not κ-metrizable. Recall that a space X is called a Hurewicz space [8] if for each sequence $\{\mathcal{U}_n\colon n \in \mathbb{N}\}$ of open covers of X, there are finite sets $\mathcal{V}_n \subset \mathcal{U}_n$ such that $\bigcup\{\mathcal{V}_n\colon n \in \mathbb{N}\}$ covers X. Arkhangel'skii proved in [2] that $C_p(X)$ has countable fan tightness iff each finite product of X is a Hurewicz space. Since it is known that the space J of irrational numbers is not a Hurewicz space [2], $C_p(J)$ does not have countable fan tightness. Hence we can find a sequence $\{A_n\colon n \in \mathbb{N}\}$ of subsets of $C_p(J)$ such that $f_0 \in \bigcap\{\bar{A}_n\colon n \in \mathbb{N}\}$, but for any finite sets $B_n \subset A_n$, $f_0 \in C_p(J) - \overline{\bigcup\{B_n\colon n \in \mathbb{N}\}}$, where f_0 is the constant function to 0. We may assume that each A_n is countable, because $C_p(J)$ has countable tightness [3, Theorem II.1.1]. Since $C_p(J)$ is separable [3, Theorem I.1.5], we can take a countable dense subset X of $C_p(J)$ which contains $\{f_0\} \cup (\bigcup\{A_n\colon n \in \mathbb{N}\})$. The space X is a countable κ-metrizable space. Since a compact space is a Hurewicz space, every EG-space must have countable fan tightness. Therefore $X(f_0)$ is not an EG-space. Thus $X(f_0)$ is not κ-metrizable by Corollary 2.6. Note that X is not monotonically normal by Fact 3.1.

4. Questions

We describe some open questions concerning EG-spaces and κ-metrizable spaces.

Question 4.1. Is every countable EG-space κ-metrizable? Equivalently, is every countable subspace of $C_p(C)$ κ-metrizable?

Question 4.2. Is every (countable) stratifiable κ-metrizable space an EG-space?

Question 4.3. Is there a universal space for all countable stratifiable κ-metrizable spaces?

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